

Holographic RG flow and Sound Modes of sQGP

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We consider the hydrodynamics of strongly interacting quark gluon plasma in finite temperature and density using the holographic duality of charged black hole in anti DeSitter space. We calculate the transport coefficients at arbitrary energy scale by considering the holographic screen at finite radial position. We first calculate the flow of sound velocity in this method and check the consistence with previous result. Then we calculate diffusion constant of charge and find that Einstein relation between susceptibility, conductivity and diffusion constants will hold at arbitrary slice.

I. INTRODUCTION

In usual application of AdS/CFT [1], physical quantities are calculated at the asymptotic AdS boundary. However, a single asymptotical AdS gravity system is expected to describe a theory both at the UV regime as well as the IR region, since the radial direction is identified as the energy scale [2–4] of the gauge theory. While the asymptotic boundary describes the UV fixed point, the horizon, as IR boundary, should be able to characterize low energy behavior of the theory. In fact, some of the transport coefficients were calculated using membrane paradigm at the horizon. Therefore it would be very interesting to find the interpolating functions connecting the UV and the IR behavior. Such interpolating functions can be computed by putting the holographic screen at a slice $r = r_c$. The paradigm to compute the slice dependent quantities has been recently proposed in [5, 6] as holographic Wilsonian renormalization group. Physical quantities at the slice have been defined in a slight different way of Wilson fluid/gravity duality in [7]. It has been shown that these two different methods, together with the earlier sliding membrane paradigm [8] are consistent [9, 10]. For further recent progress in this direction, see [11–17].

When we ask the density dependence, we need to introduce the local U(1) charge, which in turn gives an extra complexity in the hydrodynamic analysis due to the mode mixing: in the RN-AdS black hole, vector modes of gravitational fluctuations mix with transverse Maxwell modes [21, 23]. The scalar modes of metric perturbation mix with longitudinal modes of Maxwell potential and the analysis is much more involved [22]. In a companion paper [10], we focused on the former, i.e, the holographic RG flow of tensor and vector part of metric perturbations and transverse Maxwell perturbation. We proposed a systematical method to compute the hydrodynamical poles and retarded Green function at the slice $r = r_c$.

In this paper, we focus on the running of the sound modes. Notice that the running of sound modes has already been studied in [7] and in more recent paper [13]. What is new in this paper is the density effect on the sound modes coming from the mixing of gravity and Maxwell part in the presence of the charge. We will obtain analytical results for sound velocity and charge-diffusion constant and will see that they have nontrivial charge dependence in RG flows. We also find that Einstein relation between susceptibility, conductivity and diffusion constants will hold at arbitrary slice.

This paper is organized as follows. In section II, we analyze the sound part perturbations for the RN-AdS₅ and focus on the dynamical structure of Einstein and Maxwell equations. In section III, we write down the master equations which are equivalent to equations of motion in section II and solve them perturbatively in hydrodynamical limit. We impose both the in going horizon condition and Dirichlet boundary conditions at the slice surface $r = r_c$. In the following section IV we derive the on shell action for the sound fluctuations and write down forms of Green functions and pick up the pole position. We found a charge diffusion pole and a sound pole at $r = r_c$ and we discuss the RG flow of them. We found the *universal* value (charge independent) located at the horizon for both momentum diffusion and charge diffusion constant while located at the UV boundary for the speed of sound. We checked some transport coefficients and Einstein relations at the slice in the last section.

II. SOUND MODES IN REISSNER-NORDSTRÖM-ADS₅

We consider the 5 dimensional Einstein-Maxwell theory. The action is given by

$$S = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g} (R - 2\Lambda) - \frac{1}{4e^2} \int d^5x \sqrt{-g} F^2, \quad (1)$$

where five dimensional Newton constant is given by $G_5 = \kappa^2/8\pi$. The metric of the RN-AdS black hole is given by

$$ds^2 = \frac{r^2}{l^2} (-f(r)dt^2 + d\vec{x}^2) + \frac{l^2 dr^2}{r^2 f(r)}, \quad (2)$$

where

$$f(r) = 1 - \frac{l^2 m}{r^4} + \frac{l^2 q^2}{r^6}, \quad \Lambda = -\frac{6}{l^2}, \quad (3)$$

and the background gauge potential is

$$A_t = \mu - \frac{Q}{r^2}, \quad (4)$$

where

$$\frac{Q^2}{e^2} = \frac{3q^2}{2\kappa^2}. \quad (5)$$

The chemical potential μ is fixed by the regularity condition at the horizon as

$$\mu = \frac{Q}{r_+^2}. \quad (6)$$

It is convenient to introduce a coordinate $u = r_+^2/r^2$, where r_+ is the outer horizon. Then the metric is expressed as

$$ds^2 = \frac{l^2}{4b^2u} (-f(u)dt^2 + d\vec{x}^2) + \frac{l^2 du^2}{4u^2 f(u)}, \quad (7)$$

$$f(u) = (1-u)(1+u-au^2), \quad (8)$$

where

$$a = \frac{l^2 q^2}{r_+^6}, \quad b = \frac{l^2}{2r_+}. \quad (9)$$

Consider the gravitational fluctuation for the RN-AdS solution, the Gibbons-Hawking term on the boundary is necessary to well define a variation principle

$$S_{\text{GH}} = \frac{1}{\kappa^2} \int d^4x \sqrt{-\gamma} K \quad (10)$$

where $\gamma_{\mu\nu}$ and $K_{\mu\nu}$ are induced metric and extrinsic curvature on the boundary, respectively. Furthermore, in order to regularize the on shell action, we introduce the counter term [20]

$$S_{\text{ct}} = \frac{1}{\kappa^2} \int d^4x \sqrt{-\gamma} \frac{3}{l}. \quad (11)$$

From now on, we study fluctuations h_{mn} , B_m for the background metric \bar{g}_{mn} (3) and the gauge field \bar{A}_m (4),

$$g_{mn} = \bar{g}_{mn} + h_{mn}, \quad A_m = \bar{A}_m + \frac{B_m}{\mu}, \quad (12)$$

and consider the linearized theory of these fluctuations. Take the gauge condition $h_{um} = 0$ and $B_u = 0$ and consider the Fourier transforms of these fluctuation fields,

$$h_{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^4} e^{-i\omega t + ikz} h_{\mu\nu}(r), \quad (13)$$

$$B_\mu = \int \frac{d^4k}{(2\pi)^4} e^{-i\omega t + ikz} B_\mu(r), \quad (14)$$

where the momenta k is along the z -direction. The fluctuations of the metric are categorized into the tensor mode, shear mode and sound mode, and the gauge field is categorized into the vector mode and scalar mode. At the linearized order, there are mixture between the shear part of metric and transverse gauge field, also the sound mode of metric and the longitudinal gauge field. Here, we focus on the sound mode of the metric and the vector mode of the gauge field. Then, the relevant components are

$$h_t^t, \quad h_x^x = h_y^y, \quad h_z^z, \quad h_t^z, \quad B_t, \quad B_z \quad (15)$$

where we have defined $h_\nu^\mu = \bar{g}^{\mu\lambda} h_{\lambda\nu}$ for later convenience.

Relevant equations for the sound modes in the Einstein equation are (t, t) , (t, u) , (x, x) , (z, z) , (u, u) , (t, z) and (u, z) components listed as follows:

$$0 = h_t'' + \frac{(u^{-1}f)'}{u^{-1}f} \left(\frac{3}{2}h_t' + h_x' + \frac{1}{2}h_z' \right) - \frac{b^2 k^2}{uf} h_t^t + \frac{2b^2}{uf^2} \left(\omega^2 h_x^x + \frac{1}{2}\omega^2 h_z^z + \omega k h_t^z \right) + 2a \frac{u}{f} h_t^t + 4a \frac{u}{f} B_t', \quad (16a)$$

$$0 = h_x'' + \frac{(u^{-2}f)'}{u^{-2}f} h_x^{x'} - \frac{1}{2u} (h_t' + h_z') + \frac{b^2}{uf^2} (\omega^2 - k^2 f) h_x^x - a \frac{u}{f} h_t^t - 2a \frac{u}{f} B_t', \quad (16b)$$

$$0 = h_z'' + \frac{(u^{-\frac{3}{2}}f)'}{u^{-\frac{3}{2}}f} h_z^{z'} - \frac{1}{u} \left(\frac{1}{2}h_t' + h_x' \right) + \frac{b^2}{uf^2} (\omega^2 h_z^z + 2\omega k h_t^z - k^2 f h_t^t - 2k^2 f h_x^x) - a \frac{u}{f} h_t^t - 2a \frac{u}{f} B_t', \quad (16c)$$

$$0 = h_t^{z''} - \frac{1}{u} h_t^{z'} + \frac{2b^2 \omega k}{uf} h_x^x - 3au B_z', \quad (16d)$$

$$0 = h_t^{t''} + 2h_x^{x''} + h_z^{z''} + \frac{f'}{f} \left(\frac{3}{2}h_t' + h_x' + \frac{1}{2}h_z' \right) + 2a \frac{u}{f} h_t^t + 4a \frac{u}{f} B_t', \quad (16e)$$

$$0 = \omega \left[2h_x^{x'} + h_z^{z'} - \frac{f'}{f} \left(h_x^x + \frac{1}{2}h_z^z \right) \right] + k \left(h_t^{z'} - \frac{f'}{f} h_z^z \right), \quad (16f)$$

$$0 = k h_t^{t'} + 2k h_x^{x'} - \frac{\omega}{f} h_t^{z'} + \frac{k f'}{2f} h_t^t + 3a \frac{u}{f} (k B_t + \omega B_z). \quad (16g)$$

The Maxwell equations which is relevant to the vector mode are t , z , and u components:

$$0 = B_t'' - \frac{b^2}{uf} (k^2 B_t + k\omega B_z) + \frac{1}{2} (h_t^{t'} - 2h_x^{x'} - h_z^{z'}), \quad (17a)$$

$$0 = B_z'' + \frac{f'}{f} B_z' + \frac{b^2}{uf^2} (\omega^2 B_z + \omega k B_t) - \frac{1}{f} h_t^{z'}, \quad (17b)$$

$$0 = \omega B_t' + k f B_z' + \frac{\omega}{2} (h_t^t - 2h_x^x - h_z^z) - k h_t^z. \quad (17c)$$

There are 10 equations for 6 fields, and only 6 of these equations are independent. These equations are classified into the dynamical equations which are second order differential equations and first order constraints. For the Einstein equations, we have 4 dynamical equations and 3 constraint equations. Independent equations are 1 dynamical equation and 3 constraint equations. For the Maxwell equations, 1 dynamical equation and 1 constraint equation are independent. These differential equations give 8 integration constants since the solutions of n -th differential equations have n integration constants. We impose the incoming boundary condition at the horizon for each dynamical equation. Then, there remain 6 integration constants which are determined by choosing the values of the 6 fields on the UV boundary.

III. MASTER EQUATIONS AND BOUNDARY CONDITIONS

A. Master equations

Consider the two independent dynamical equations. By using an appropriate redefinition of the fields, we obtain the dynamical equations which always satisfy the constraint equations. Such fields and equations are called master fields and master equations, respectively. The master fields are derived in [21], and can be expressed in our notation as

$$\Phi = \frac{1}{4u^{3/2}(4b^2k^2 - 3f')} [(4b^2k^2 - 3f')h_x^x + 2f(h_x^{x'} + h_z^{z'})], \quad (18)$$

for metric perturbations and

$$\mathcal{A} = 2a(-h_t^t + 3h_x^x - 2B_t'), \quad (19)$$

for gauge part. Two master equations for the master fields (18) and (19) can be diagonalized by the following new master fields:

$$\Phi_{\pm} = \alpha_{\pm}\Phi + \frac{u^{1/4}}{8}\mathcal{A}, \quad (20)$$

where

$$\alpha_{\pm} = (1+a)(1 \pm \alpha) - 3au_c, \quad (21)$$

$$\alpha = \sqrt{1 + \frac{4ab^2k^2}{(1+a)^2}}. \quad (22)$$

The final master equations become

$$\Phi_{\pm}'' + \frac{(u^{1/2}f)'}{u^{1/2}f}\Phi_{\pm}' + V_{\pm}\Phi_{\pm} = 0, \quad (23)$$

where potentials V_{\pm} are given by

$$\begin{aligned} V_{\pm} = & \frac{1}{16u^2f^2(4b^2k^2 - 3f')^2} \times \\ & \times \left\{ -4uf(4b^2k^2 - 3f') \left(16b^2k^2(b^2k^2 - C_{\pm}u + 3au^2) \right. \right. \\ & \quad \left. \left. - 4f'(2b^2k^2 + 3C_{\pm}u - 9au^2) - 3(f')^2 \right) \right. \\ & \quad \left. + f^2 \left[16 \left(b^4k^4 + 12C_{\pm}b^2k^2u - 108ab^2k^2u^2 + 54C_{\pm}au^3 - 162a^2u^4 \right) \right. \right. \\ & \quad \left. \left. - 24f' \left(b^2k^2 - 6C_{\pm}u - 18au^2 \right) + 9(f')^2 \right] \right\} + \frac{b^2\omega^2}{uf^2}. \end{aligned} \quad (24)$$

Let us solve these master equations. We first impose the incoming boundary condition at the horizon. Near the horizon $u = 1$, the solutions of the master equation (23) behave as

$$\Phi_{\pm} \sim (1 - u)^{\nu}, \quad (25)$$

where there are two independent solutions with $\nu = \pm i b \omega / (2 - a)$. Each of them corresponds to the incoming and outgoing modes at the horizon. We impose the incoming boundary condition by taking only $\nu = -i b \omega / (2 - a)$. Then, the solution can be factorized as

$$\Phi_{\pm} = (1 - u)^{-i b \omega / (2 - a)} F_{\pm}. \quad (26)$$

Since we have factorized the most singular part near the horizon, F_{\pm} should be regular at the horizon. In the hydrodynamic region, ω and k are small and we can solve the master equations order by order by expanding the master fields with respect to ω and k . It was solved in [22], and the solutions take the form of

$$\Phi_{\pm} = C_{\pm} H_{\pm}(u) (1 - u)^{-i b \omega / (2 - a)} \left(1 + b \omega F_{\pm 10}(u) + b^2 \omega^2 F_{\pm 20}(u) + b^2 k^2 F_{\pm 02}(u) + \dots \right). \quad (27)$$

where H_{\pm} given by

$$H_{+} = u^{-3/4}, \quad H_{-} = \frac{u^{1/4}}{(1 + a) - \frac{3}{2} a u}, \quad (28)$$

are factorized such that the leading terms in F_{\pm} become u independent constant. The determination of integration constants C_{\pm} is important for computing the final Green functions and we will see it soon. Details of $F_{\pm ij}$ are shown in Appendix A.

B. Boundary Conditions at Slice $u = u_c$

In order to determine the integration constant C_{\pm} we need boundary conditions of the original fields h_{ν}^{μ} and B_{μ} . Consider the boundary at slice $u = u_c$ and impose the boundary values for the original fields as

$$h_{\nu}^{\mu}(u_c) = h_{\nu}^{\mu(0)}, \quad B_{\mu}(u_c) = B_{\mu}^{(0)}. \quad (29)$$

Taking use of the Einstein equations and Maxwell equations at the slice, we obtain the following relations between the integration constants and boundary conditions:

$$\mathcal{E}_{+} C_{+} + \mathcal{E}_{-} C_{-} = \mathcal{E}_0, \quad (30)$$

$$\mathcal{M}_{+} C_{+} + \mathcal{M}_{-} C_{-} = \mathcal{M}_0, \quad (31)$$

where \mathcal{E}_0 and \mathcal{M}_0 are linear combinations of $h_{\nu}^{\mu(0)}$ and $(B_{\mu})^{(0)}$. Namely they take the form of

$$\mathcal{E}_0 = \mathcal{E}_{0I} \phi_I^{(0)}, \quad \mathcal{M}_0 = \mathcal{M}_{0I} \phi_I^{(0)}, \quad (32)$$

where ϕ_I stands for the original fields, h_ν^μ or B_μ , and $\mathcal{E}_{0I}, \mathcal{M}_{0I}$ are given functions of ω, k, a, b and u_c . \mathcal{E}_\pm and \mathcal{M}_\pm are also known functions of ω, k, a, b, u_c and generally they take the form of

$$\mathcal{E}_\pm = \mathcal{E}_{\pm 0} \tilde{F}_\pm(u_c) + \mathcal{E}_{\pm 1} \tilde{F}'_\pm(u_c), \quad (33)$$

$$\mathcal{M}_\pm = \mathcal{M}_{\pm 0} \tilde{F}_\pm(u_c) + \mathcal{M}_{\pm 1} \tilde{F}'_\pm(u_c), \quad (34)$$

where $\mathcal{E}_{\pm 0}, \mathcal{E}_{\pm 1}, \mathcal{M}_{\pm 0}, \mathcal{M}_{\pm 1}$ are all known coefficients and \tilde{F}_\pm are defined in terms of the solutions of the master fields as

$$\tilde{F}_\pm(u) = (1-u)^{-ib\omega/(2-a)} \left(1 + b\omega F_{\pm 10}(u) + b^2\omega^2 F_{\pm 20}(u) + b^2k^2 F_{\pm 02}(u) + \dots \right). \quad (35)$$

Remember $\tilde{F}_\pm(u)$ are given functions since we have solved the master equations. Once the above ingredients are given, C_\pm can be finally expressed in terms of the boundary conditions as

$$C_+ = \frac{\mathcal{E}_0 \mathcal{M}_- - \mathcal{M}_0 \mathcal{E}_-}{\mathcal{E}_+ \mathcal{M}_- - \mathcal{M}_+ \mathcal{E}_-}, \quad C_- = \frac{\mathcal{M}_0 \mathcal{E}_+ - \mathcal{E}_0 \mathcal{M}_+}{\mathcal{E}_+ \mathcal{M}_- - \mathcal{M}_+ \mathcal{E}_-}. \quad (36)$$

IV. POLE OF GREEN FUNCTIONS AT THE SLICE $u = u_c$

Now we consider Green functions. By using the equations of motions, the on-shell action for fluctuations is reduced to surface terms as

$$\begin{aligned} S = \frac{l^3}{32\kappa^2 b^4} \int \frac{d^4 k}{(2\pi)^4} \left\{ \frac{1}{u} h_t^z h_t^{z'} + \frac{f}{u} h_x^x h_x^{x'} + \frac{f}{u} h_t^t h_x^{x'} + \frac{f}{2u} h_t^t h_z^{z'} \right. \\ + \frac{f}{u} h_x^x h_t^{t'} + \frac{f}{u} h_x^x h_z^{z'} + \frac{f}{2u} h_z^z h_t^{t'} + \frac{f}{u} h_z^z h_x^{x'} \\ + \frac{3}{4u^2} (f - \sqrt{f}) (h_t^t)^2 + \frac{1}{4u^2} (3f - uf' - 3\sqrt{f}) (h_z^z)^2 \\ - \frac{3}{u^2 f} (f - \sqrt{f}) (h_t^z)^2 - \frac{1}{2u^2} (6f - uf' - 6\sqrt{f}) h_t^t h_x^x \\ - \frac{1}{4u^2} (6f - uf' - 6\sqrt{f}) h_t^t h_z^z - \frac{1}{u^2} (3f - uf' - 3\sqrt{f}) h_x^x h_z^z \\ \left. + 3a \left(B_t B_t' - f B_z B_z' + \frac{1}{2} B_t h_t^t + B_z h_t^z - B_t h_x^x - \frac{1}{2} B_t h_z^z \right) \right\} \Big|_{u=u_c}. \quad (37) \end{aligned}$$

It can be rewritten as

$$S = \int \frac{dk^4}{(2\pi)^2} \left(\phi_I(u_c) \mathcal{K}_{IJ} \phi_J(u_c) + \phi_I(u_c) \mathcal{L}_{IJ} \phi_J'(u_c) \right). \quad (38)$$

By using the constraint equations and the definitions of the master fields, we obtain the relation between the original fields ϕ_I and its first order derivatives ϕ_I' . Since we have 4

constraint equations and 2 master fields, we can diagonalize these equations such that the 6 first derivatives ϕ'_I at the boundary can be expressed in terms of the boundary values $\phi_I^{(0)}$. They take the following form:

$$\phi'_I = \mathcal{H}_{IJ}\phi_J + \mathcal{H}_{I+}C_+\tilde{F}_+ + \mathcal{H}_{I-}C_-\tilde{F}_- . \quad (39)$$

Combined with (38) the on shell action is obtained as

$$S = \int \frac{dk^4}{(2\pi)^4} \phi_I^{(0)} \mathcal{G}_{IJ} \phi_J^{(0)} , \quad (40)$$

where \mathcal{G}_{IJ} is given by

$$\begin{aligned} \mathcal{G}_{IJ} = \mathcal{K}_{IJ} + \mathcal{L}_{IK} & \left(\mathcal{H}_{KJ} + \mathcal{H}_{K+}\tilde{F}_+(u_c) \frac{\mathcal{E}_{0J}\mathcal{M}_- - \mathcal{M}_{0J}\mathcal{E}_-}{\mathcal{E}_+\mathcal{M}_- - \mathcal{M}_+\mathcal{E}_-} \right. \\ & \left. + \mathcal{H}_{K-}\tilde{F}_-(u_c) \frac{\mathcal{M}_{0J}\mathcal{E}_+ - \mathcal{E}_{0J}\mathcal{M}_+}{\mathcal{E}_+\mathcal{M}_- - \mathcal{M}_+\mathcal{E}_-} \right) . \end{aligned} \quad (41)$$

Finally the retarded Green functions at the slice G^R are given by [19]

$$G_{IJ}^R = 2\mathcal{G}_{IJ} . \quad (42)$$

We find the pole of the Green functions is located at

$$\mathcal{E}_+\mathcal{M}_- - \mathcal{E}_-\mathcal{M}_+ = 0 . \quad (43)$$

For convenience we rewrite (35) as

$$\tilde{F}_\pm(u) = 1 + b\omega\tilde{F}_{\pm 10}(u) + b^2\omega^2\tilde{F}_{\pm 20}(u) + b^2k^2\tilde{F}_{\pm 02}(u) + \dots . \quad (44)$$

The pole has the following structure:

$$0 = P_{40}\omega^4 + P_{30}\omega^3 + P_{22}\omega^2k^2 + P_{12}\omega k^2 + P_{04}k^4 + \dots , \quad (45)$$

where the coefficients are listed as follows

$$P_{40} = 9b^2f'(u_c) \left[\tilde{F}_{+10}\tilde{F}'_{-10} + \tilde{F}'_{-20}(u_c) \right] , \quad (46)$$

$$P_{30} = 9bf'(u_c)\tilde{F}'_{-10}(u_c) , \quad (47)$$

$$P_{22} = \left\{ -3b^2 \left[f(u_c)(f'(u_c) + 6au_c^2) - u_cf'^2(u_c) \right] \left[\tilde{F}_{+10}\tilde{F}'_{-10} + \tilde{F}'_{-20}(u_c) \right] \right. \quad (48)$$

$$\left. + 9b^2f'(u_c)\tilde{F}'_{-02}(u) - 12b^2f^2(u_c)\tilde{F}'_{+10}(u_c)\tilde{F}'_{-10}(u_c) + \frac{54a^2b^2u_c^2}{(1+a)f'(u_c)} \right\} , \quad (49)$$

$$P_{12} = -3b \left[f(u_c)(f'(u_c) + 6au_c^2) - u_cf'^2(u_c) \right] \tilde{F}'_{-10}(u_c) , \quad (50)$$

$$P_{04} = \left\{ -3b^2 \left[f(u_c)(f'(u_c) + 6au_c^2) - u_cf'^2(u_c) \right] \tilde{F}'_{-02}(u_c) + \frac{6ab^2u_c^3}{(1+a)f'^2(u_c)} \times \right. \quad (51)$$

$$\left. \left[6a(1+a) + (8+24a-3a^2+8a^3)u_c - 18a(1+a)^2u_c^2 + 15a^2(1+a)u_c^3 \right] \right\} . \quad (52)$$

Apparently P_{ij} coefficients here can only be fixed up to an overall constant and we leave the explicit form of $\mathcal{E}_+\mathcal{M}_- - \mathcal{E}_-\mathcal{M}_+$ to the Appendix B. Due to the structure (45), the pole has 4 roots which are referred to as

$$\omega = 1/\tau + \dots, \quad \omega = \pm c_s k + \dots, \quad \omega = -iD_A k^2 + \dots, \quad (53)$$

and can be factorized to be product of these roots:

$$0 = (\omega - c_s k + \dots)(\omega + c_s k + \dots)(\omega + iD_A k^2 + \dots)(\omega - 1/\tau + \dots) \quad (54)$$

$$= -\frac{1}{\tau}\omega^3 + \omega^4 + \frac{c_s^2}{\tau}\omega k^2 - \left(c_s^2 - i\frac{D_A}{\tau}\right)\omega^2 k^2 - i\frac{c_s^2 D_A}{\tau}k^4 + \dots \quad (55)$$

Here “...” denote the higher derivative corrections. By matching (45) and (55), we obtain the bare sound speed

$$c_s^2 = -\frac{P_{12}}{P_{30}} = \frac{f(u_c)[f'(u_c) + 6au_c^2] - u_c f'^2(u_c)}{3f'(u_c)}, \quad (56)$$

and the bare charge diffusion constant

$$D_A = -i\frac{P_{04}}{P_{12}} = \text{Im} \left[b \frac{\tilde{F}'_{-02}(u_c)}{\tilde{F}'_{-10}(u_c)} - b \frac{2au_c^3[6a(1+a) + (8+24a-3a^2+8a^3)u_c - 18a(1+a)^2u_c^2 + 15a^2(1+a)u_c^3]}{(1+a)f'^2(u_c)[f(u_c)(f'(u_c) + 6au_c^2) - u_c f'^2(u_c)]\tilde{F}'_{-10}(u_c)} \right]. \quad (57)$$

In the orthonormal frame, the proper frequency ω_c and proper momentum k_c are defined by

$$\omega_c = \frac{\omega}{\sqrt{-g_{tt}}}, \quad k_c = \frac{k}{\sqrt{g_{zz}}}. \quad (58)$$

The normalized speed of sound and the diffusion constant are given by

$$\bar{c}_s^2 = -\frac{g_{zz}}{g_{tt}}c_s^2 = \frac{f(u_c)[f'(u_c) + 6au_c^2] - u_c f'^2(u_c)}{3f(u_c)f'(u_c)}, \quad (59)$$

$$\bar{D}_A = -\frac{g_{zz}}{g_{tt}}T_H D_A = \text{Im} \left[\frac{2-a}{4\pi} \frac{\tilde{F}'_{-02}(u_c)}{f(u_c)\tilde{F}'_{-10}(u_c)} - \frac{a(2-a)u_c^3[6a(1+a) + (8+24a-3a^2+8a^3)u_c - 18a(1+a)^2u_c^2 + 15a^2(1+a)u_c^3]}{2\pi(1+a)f(u_c)f'^2(u_c)[f(u_c)(f'(u_c) + 6au_c^2) - u_c f'^2(u_c)]\tilde{F}'_{-10}(u_c)} \right]. \quad (60)$$

V. RG FLOWS OF SOUND SPEED AND DIFFUSION CONSTANTS

We will summarize radial flows of various quantities in this section. It is shown in Left part of Fig 1 that the bare speed of sound flows quite differently for different charges. For

zero charge case, $c_s^2(u)$ runs from $\frac{1}{3}$ to $\frac{2}{3}$ if we move from UV boundary to IR horizon. This is consistent with the chargeless result in [7, 13] at the horizon and also at the UV boundary [18]. As suggested in [7], the physical speed of sound may be defined as the one in the orthonormal frame, which is shown in right part of Fig 1. The normalized sound speed always diverges at the horizon for any charge. One interesting observation is that both for bare one or normalized one, there is a universal UV boundary value of sound speed $\frac{1}{\sqrt{3}}$ for different charges. Let us turn to the diffusion. Remember the momentum diffusion D_h is encoded in hydrodynamical pole in shear part of metric fluctuations [10] and charge diffusion D_A is contained in sound part. As shown in Fig 2, both for normalized \bar{D}_h and \bar{D}_A , the universal charge independent value $\frac{1}{4\pi}$ is located at the horizon.

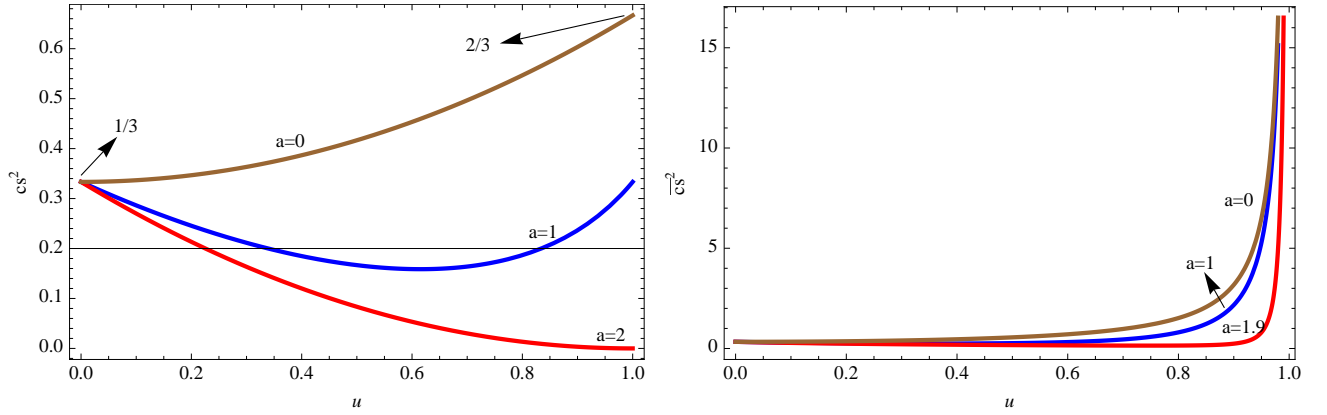


FIG. 1: Radial flow of speed of sound c_s^2 . Left: The bare speed of sound. Right: Normalized speed of sound in orthonormal frame.

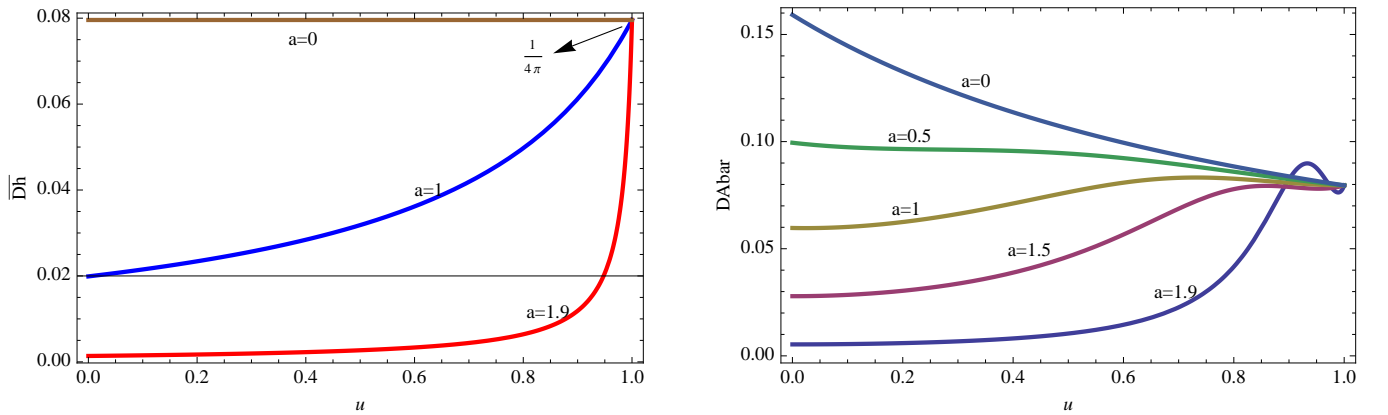


FIG. 2: Radial flow of diffusion constants. Left: The momentum diffusion constant coming from shear part hydrodynamical pole [10]. Right: Charge diffusion constant coming from sound part hydrodynamical pole.

VI. TRANSPORTS AND EINSTEIN RELATION AT THE SLICE r_c

Retarded Green functions can be computed explicitly following (41). Generally the Green functions have complicated expressions. Rather than showing all the explicit expressions we focus on G_{zz} , G_{ijjj} and G_{tt} at certain limits. By using the Kubo formula, we obtain

$$\begin{aligned}\sigma &= \lim_{\omega \rightarrow 0} \frac{1}{\omega} \text{Im} G_{zz}(\omega, k=0) = \frac{l^3}{16\kappa^2 b^4 \mu^2} \frac{3ab}{2u_c f(u_c) \text{Im} \tilde{F}'_{-10}(u_c)}, \\ &= \frac{l}{2e^2 b u_c f(u_c) \text{Im} \tilde{F}'_{-10}(u_c)}.\end{aligned}\quad (61)$$

This is electrical conductivity, which is the same as our previous result [10]. We also find that the bulk viscosity vanishes at arbitrary slice $u = u_c$

$$\zeta = \lim_{\omega \rightarrow 0} \frac{1}{\omega} \sum_{i,j=x,y,z} \text{Im} G_{ijjj}(\omega, k=0) = 0. \quad (62)$$

In $k \rightarrow 0$ limit, the density-density correlation function G_{tt} behaves as

$$\lim_{k \rightarrow 0} \frac{\text{Im} G_{tt}}{k^2} = \frac{l^3}{16b^4 \kappa^2 \mu^2} \frac{3ab}{2\omega u_c f(u_c) \text{Im} \tilde{F}'_{-10}(u_c)}. \quad (63)$$

Together with the pole structure which we have discussed above, it implies that

$$\text{Im} G_{tt}(\omega, k) \sim \frac{l^3}{16b^4 \kappa^2 \mu^2} \frac{3ab}{2u_c f(u_c) \text{Im} \tilde{F}'_{-10}(u_c)} \frac{k^2 \omega^3}{(\omega^2 + D_A^2 k^4)(\omega^2 - c_s^2 k^2)} + \dots \quad (64)$$

and then we obtain

$$\text{Im} G_{tt}(\omega, k=0) = \lim_{k \rightarrow 0} \text{Im} G_{tt}(\omega, k) = \frac{l}{2e^2 b} \frac{-\omega}{u_c f(u_c) \text{Im} \tilde{F}'_{-10}(u_c)} \frac{2\pi}{D_A} \delta(\omega). \quad (65)$$

By using the definition of the charge susceptibility

$$\Xi = -\frac{1}{T} \int \frac{d\omega}{2\pi} \frac{\text{Im} G_{tt}(\omega, k=0)}{e^{\omega/T} - 1}, \quad (66)$$

we obtain

$$\Xi = \frac{1}{D_A} \frac{l}{2e^2 b u_c f(u_c) \text{Im} \tilde{F}'_{-10}(u_c)}, \quad (67)$$

which satisfies the Einstein relation

$$\Xi = \frac{\sigma}{D_A}. \quad (68)$$

As a summary, for the transport coefficients, the shear viscosity keeps a constant from the horizon to the UV boundary and DC electric conductivity has a nontrivial radial flow as explicitly shown in [10]. Here we checked that the bulk viscosity vanishes at arbitrary slice and the Einstein relation holds at the slice.

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Appendix A: Concrete expressions

The solutions of F_{\pm} are the following:

$$F_{+10}(u) = \frac{i}{2(2-a)} \left\{ \log(1+u-au^2) - \frac{6K_1(u)}{\sqrt{1+4a}} \right\}, \quad (\text{A1a})$$

$$F_{+02}(u) = \frac{2}{3} \left\{ \frac{K_1(u)}{\sqrt{1+4a}} - \frac{1}{(1+a)u} \right\}, \quad (\text{A1b})$$

$$F_{+20}(u) = \int^u du \frac{1}{(1-u)(1+u-au^2)} \times \left\{ 1-u + \frac{(1-u)(1+au) \log(1+u-au^2)}{2(2-a)^2} - \frac{3(1-u)(1+au)K_2(0)}{2(2-a)^2\sqrt{1+4a}} \right. \\ \left. - \frac{(1+a)K_2(1)u}{\sqrt{1+4a}} + \frac{(3+(5+3a-6a^2+2a^3)u-3au^2)K_2(u)}{2(2-a)^2\sqrt{1+4a}} \right\}, \quad (\text{A1c})$$

$$F_{-10}(u) = \frac{i}{2(2-a)^2} \left\{ 8(1+a)^2 \log(u) - (2+a)(1+4a) \log(1+u-au^2) \right. \\ \left. - 2\sqrt{1+4a}(2+5a)K_1(u) \right\}, \quad (\text{A1d})$$

$$F_{-02}(u) = \left\{ -\frac{3a^2u}{2(1+a)^2\left(1+a-\frac{3}{2}au\right)} - \frac{2(1+a)(2+a) \log(u)}{(2-a)^2} \right. \\ \left. + \frac{(1+a)(2+a) \log(1+u-au^2)}{(2-a)^2} + \frac{2(2+5a+6a^2)K_1(u)}{(2-a)^2\sqrt{1+4a}} \right\}, \quad (\text{A1e})$$

$$F_{-20}(u) = \int^u du \frac{1}{2(2-a)^4(1+4a)^{3/2}(1-u)u(1+u-au^2)} \times \left\{ 8(2-a)(1+a)^2(1+4a)^{3/2}u(1+u-au^2) \log(u) \right. \\ - (2-a)(1+4a)^{3/2} \left(4(1+a)^2 + (2-3a-8a^2)u + (2+9a+13a^2)u^2 \right. \\ \left. - a(2+a)(1+4a)u^3 \right) \log(1+u-au^2) \\ + (1+4a)^2(2+5a)K_2(0)(1-u) \left(4(1+a)^2 + (2-3a-8a^2)u - (2-a)au^2 \right) \\ - 2(1+a)(2-2a+41a^2)K_2(1) \left(2(1+a) - 3au \right)^2 \\ - (2-a) \left(a(1+4a)^2(2+5a)u^3 - (2-a)(1+11a+46a^2+18a^3)u^2 \right. \\ \left. - (2+9a+180a^2+224a^3+24a^4)u \right. \\ \left. - 4(1+a)^2(1-10a-2a^2) \right) K_2(u) \Big\}. \quad (\text{A1f})$$

where

$$K_1(u) = \frac{1}{2} \log(1 + u - au^2) - \log\left(1 - \frac{2au}{1 + \sqrt{1 + 4a}}\right), \quad (\text{A2})$$

$$K_2(u) = \log\left(\frac{1 + \sqrt{1 + 4a} - 2au}{-1 + \sqrt{1 + 4a} + 2au}\right). \quad (\text{A3})$$

Then, \tilde{F}_\pm are related to these solutions as

$$\tilde{F}_{\pm 10}(u) = F_{\pm 10} - \frac{i}{2-a} \log(1-u) \quad (\text{A4})$$

$$\tilde{F}_{\pm 20}(u) = F_{\pm 20}(u) - \frac{i}{2-a} \log(1-u) F_{\pm 10}(u) - \frac{1}{2(2-a)^2} \log^2(1-u) \quad (\text{A5})$$

$$\tilde{F}_{\pm 02}(u) = F_{\pm 02}(u) \quad (\text{A6})$$

\mathcal{E}_{0I} and \mathcal{M}_{0I} are defined as coefficients of the original fields in

$$\mathcal{E}_0 = \mathcal{E}_{0I} \phi_I^{(0)}, \quad \mathcal{M}_0 = \mathcal{M}_{0I} \phi_I^{(0)}. \quad (\text{A7})$$

They can be read off from the expressions of \mathcal{E}_0 and \mathcal{M}_0 :

$$\begin{aligned} \mathcal{E}_0 = \frac{1}{24k^2 u_c f^3(u_c)} & \left\{ -k u_c f(u_c) (4b^2 k^2 - 3f'(u_c)) \left[-4b^2 k^3 (-h_t^{t(0)} + h_x^{x(0)}) \right. \right. \\ & \left. - 18a u_c (k B_t^{(0)} - k u_c h_x^{x(0)} + \omega B_z^{(0)}) + 3k f'(u_c) (-h_t^{t(0)} + h_x^{x(0)}) \right] \\ & \left. + u_c (4b^2 k^2 - 3f'(u_c))^2 \left[2\omega k h_t^{z(0)} - (\omega^2 + k^2 u_c f'(u_c)) h_x^{x(0)} + \omega^2 h_z^{z(0)} \right] \right\}, \quad (\text{A8}) \end{aligned}$$

$$\begin{aligned} \mathcal{M}_0 = \frac{1}{8k^2 u_c f^2(u_c)} & \left\{ k f(u_c) \left[-18a u_c^2 (k B_t^{(0)} - k u_c h_x^{x(0)} + \omega B_z^{(0)}) \right. \right. \\ & \left. - 4b^2 k^2 (2k B_t^{(0)} - k u_c h_t^{t(0)} - k u_c h_x^{x(0)} + 2\omega B_z^{(0)}) + 3k u_c f'(u_c) (h_x^{x(0)} - h_t^{t(0)}) \right] \\ & \left. + u (4b^2 k^2 - 3f'(u_c)) \left[-2\omega k h_t^{z(0)} + (\omega^2 + k^2 u_c f'(u_c)) h_x^{x(0)} - \omega^2 h_z^{z(0)} \right] \right\}. \quad (\text{A9}) \end{aligned}$$

\mathcal{E}_\pm and \mathcal{M}_\pm contain $\tilde{F}_\pm(u_c)$ and their derivatives. They are referred to as $\mathcal{E}_{\pm 0}$, $\mathcal{E}_{\pm 1}$, $\mathcal{M}_{\pm 0}$

and $\mathcal{M}_{\pm 1}$, respectively, and obtained as

$$\mathcal{E}_{+0} = \frac{1}{12(1+a)\alpha u_c f^3(u_c) k^2} \left\{ -3k^2 u_c f(u_c) (4b^2 k^2 - 3f'(u_c)) (4b^2 k^2 + 2u_c \alpha_- - f'(u_c)) \right. \\ \left. + u_c (4b^2 k^2 - 3f'(u_c))^2 (3\omega^2 + u_c f'(u_c) k^2) \right. \\ \left. - 12k^2 f^2(u_c) (4b^2 k^2 + 3(1+a)(1-\alpha)u_c) \right\} \quad (\text{A10})$$

$$\mathcal{E}_{+1} = \frac{4b^2 k^2 - 3f'(u_c)}{(1+a)\alpha f(u_c)} \quad (\text{A11})$$

$$\mathcal{E}_{-0} = \frac{u_c}{6(1+a)\alpha k^2 f^3(u_c) f'^2(u_c)} \times \\ \times \left\{ 2k^2 f'^2(u_c) \left[(3u_c f(u_c) (3\alpha_+ u_c + 8b^2 k^2) - 18f^2(u_c)) + 4b^2 u_c (2b^2 k^4 u_c - 9\omega^2) \right] \right. \\ \left. - 12k^2 f'(u_c) \left[2b^2 k^2 u_c f(u_c) (\alpha_+ u_c + 2b^2 k^2) + f^2(u_c) (3\alpha_+ u_c - 4b^2 k^2) - 4b^4 k^2 \omega^2 u_c \right] \right. \\ \left. - 48b^2 k^4 u_c f^2(u_c) f''(u_c) - 3u_c f'^3(u_c) (8b^2 k^4 u_c + 3k^2 f(u_c) - 9\omega^2) + 9k^2 u_c^2 f'^4(u_c) \right\} \quad (\text{A12})$$

$$\mathcal{E}_{-1} = \frac{2u_c^2 (4b^2 k^2 - 3f'(u_c))}{(1+a)\alpha f(u_c) f'(u_c)} \quad (\text{A13})$$

$$\mathcal{M}_{+0} = -\frac{1}{4a(1+a)\alpha u_c^2 k^2 f^2(u_c)} \left\{ 4(1+a)(1-\alpha) f^2(u_c) k^2 \right. \\ \left. + a u_c^2 f(u_c) k^2 (-4b^2 k^2 - 6u_c \alpha_- + 3f'(u_c)) \right. \\ \left. + a u_c^2 (4b^2 k^2 - 3f'(u_c)) (3\omega^2 + u_c f'(u_c) k^2) \right\} \quad (\text{A14})$$

$$\mathcal{M}_{+1} = \frac{\alpha_-}{a(1+a)u_c \alpha} \quad (\text{A15})$$

$$\mathcal{M}_{-0} = \frac{1}{2a(1+a)\alpha k^2 f'^2(u_c) f^2(u_c)} \left\{ -4k^2 \alpha_+ u_c f''(u_c) f^2(u_c) \right. \\ \left. + 2k^2 f'(u_c) \left[-6a u_c^2 b^2 \omega^2 + a u_c^2 (2b^2 k^2 + 3u_c \alpha_+) f(u_c) + 2(\alpha_+ - 3a u_c) f^2(u_c) \right] \right. \\ \left. - a u_c^2 f'^2(u_c) [4u_c b^2 k^4 - 9\omega^2 + 3k^2 (f(u_c) - u_c f'(u_c))] \right\} \quad (\text{A16})$$

$$\mathcal{M}_{-1} = \frac{2\alpha_+ u_c}{a(1+a)\alpha f'(u_c)}. \quad (\text{A17})$$

The first derivatives of original fields are expressed as

$$\phi'_I = \mathcal{H}_{IJ} \phi_J + \mathcal{H}_{I+} C_+ \tilde{F}_+ + \mathcal{H}_{I-} C_- \tilde{F}_-, \quad (\text{A18})$$

whose explicit forms are as follows:

$$\begin{aligned}
h_t^{t'} = & -\frac{2b^2k^2}{3f}h_t^t + \frac{2b^2\omega^2}{3f^2}h_z^z + \frac{4b^2k\omega}{3f^2}h_t^z \\
& + \frac{f(-18au^2 + 4b^2k^2 - 9f') - 4b^2k^2uf' + 8b^2\omega^2 + 3uf'^2}{6f^2}h_x^x \\
& - \frac{3f(4b^2k^2 - 3f' + 2u\alpha_-) + uf'(3f' - 4b^2k^2)}{3(a+1)\alpha f^2}C_+\tilde{F}_+ \\
& - \frac{2u^2(3f(4b^2k^2 - 3f' + 2u\alpha_+) + uf'(3f' - 4b^2k^2))}{3(a+1)\alpha f^2 f'}C_-\tilde{F}_-
\end{aligned} \tag{A19}$$

$$\begin{aligned}
h_x^{x'} = & \frac{4b^2k^2 - 3f'}{12f}h_t^t + \frac{\omega^2(3f' - 4b^2k^2)}{12f^2k^2}h_z^z + \frac{3\omega f' - 4b^2k^2\omega}{6f^2k}h_t^z \\
& + \frac{fk^2(18au^2 - 4b^2k^2 + 9f') + (4b^2k^2 - 3f')(k^2uf' + \omega^2)}{12f^2k^2}h_x^x - \frac{3au}{2f}B_t - \frac{3au\omega}{2fk}B_z \\
& - \frac{(4b^2k^2 - 3f')(k^2uf' + 3\omega^2) - 3fk^2(4b^2k^2 - 3f' + 2\alpha_-u)}{6(a+1)\alpha f^2k^2}C_+\tilde{F}_+ \\
& - \frac{u^2((4b^2k^2 - 3f')(k^2uf' + 3\omega^2) - 3fk^2(4b^2k^2 - 3f' + 2\alpha_+u))}{3(a+1)\alpha f^2k^2 f'}C_-\tilde{F}_-
\end{aligned} \tag{A20}$$

$$\begin{aligned}
h_z^{z'} = & \frac{3f' - 4b^2k^2}{6f}h_t^t + \frac{\omega^2(4b^2k^2 - 3f')}{6f^2k^2}h_z^z + \frac{\omega(4b^2k^2 - 3f')}{3f^2k}h_t^z \\
& - \frac{2f(9ak^2u^2 + 4b^2k^4) + (4b^2k^2 - 3f')(k^2uf' + \omega^2)}{6f^2k^2}h_x^x + \frac{3au}{f}B_t + \frac{3au\omega}{fk}B_z \\
& + \frac{(4b^2k^2 - 3f')(k^2uf' + 3\omega^2) - 6\alpha_-fk^2u}{3(a+1)\alpha f^2k^2}C_+\tilde{F}_+ \\
& + \frac{2u^2((4b^2k^2 - 3f')(k^2uf' + 3\omega^2) - 6\alpha_+fk^2u)}{3(a+1)\alpha f^2k^2 f'}C_-\tilde{F}_-
\end{aligned} \tag{A21}$$

$$\begin{aligned}
h_t^{z'} = & \frac{\omega(4b^2k^2 - f')}{2fk}h_x^x + \frac{\omega f'}{2fk}h_z^z + \frac{f'}{f}h_t^z \\
& - \frac{4b^2k^2\omega - 3\omega f'}{a\alpha fk + \alpha fk}C_+\tilde{F}_+ - \frac{2u^2\omega(4b^2k^2 - 3f')}{(a+1)\alpha f k f'}C_-\tilde{F}_-
\end{aligned} \tag{A22}$$

$$B_t' = -\frac{1}{2}h_t^t + \frac{3}{2}h_x^x + \frac{\alpha_-}{a(a+1)\alpha u}C_+\tilde{F}_+ + \frac{2u^2\alpha_+}{a(a+1)\alpha f'}C_-\tilde{F}_- \tag{A23}$$

$$B_z' = -\frac{\omega}{2fk}h_x^x + \frac{\omega}{2fk}h_z^z + \frac{1}{f}h_t^z - \frac{\alpha_- \omega}{a(a+1)\alpha f k u}C_+\tilde{F}_+ - \frac{2\alpha_+ u \omega}{a(a+1)\alpha k f f'}C_-\tilde{F}_- \tag{A24}$$

Appendix B: Pole Structure

The pole is expanded as

$$\begin{aligned}
& \frac{u_c}{a(1+a)f^3(u_c)} \left\{ 9bf'(u_c)\tilde{F}'_{-10}(u_c)\omega^3 + 9b^2f'(u_c) \left[\tilde{F}_{+10}\tilde{F}'_{-10} + \tilde{F}'_{-20}(u_c) \right] \omega^4 \right. \\
& \quad - 3b \left[f(u_c)(f'(u_c) + 6au_c^2) - u_cf'^2(u_c) \right] \tilde{F}'_{-10}(u_c)\omega k^2 \\
& \quad + \left\{ -3b^2 \left[f(u_c)(f'(u_c) + 6au_c^2) - u_cf'^2(u_c) \right] \left[\tilde{F}_{+10}\tilde{F}'_{-10} + \tilde{F}'_{-20}(u_c) \right] \right. \\
& \quad \quad + 9b^2f'(u_c)\tilde{F}'_{-02}(u) - 12b^2f^2(u_c)\tilde{F}'_{+10}(u_c)\tilde{F}'_{-10}(u_c) \\
& \quad \quad \left. \left. + \frac{54a^2b^2u_c^2}{(1+a)f'(u_c)} \right\} \omega^2 k^2 \right. \\
& \quad + \left\{ -3b^2 \left[f(u_c)(f'(u_c) + 6au_c^2) - u_cf'^2(u_c) \right] \tilde{F}'_{-02}(u_c) \right. \\
& \quad \quad + \frac{6ab^2u_c^3}{(1+a)f'^2(u_c)} \left[6a(1+a) + (8+24a-3a^2+8a^3)u_c \right. \\
& \quad \quad \left. \left. \left. - 18(a(1+a)^2)u_c^2 + 15a^2(1+a)u_c^3 \right] \right\} k^4 + \dots \right\} \\
& = 0.
\end{aligned} \tag{B1}$$

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